UNCLASSIFIED
SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER: AFOSR-TR-77-0995	2. GOVT ACCESSION NO.
3. RECIPIENT'S CATALOG NUMBER	
4. TITLE (and Subtitle): THE SLIDING OF A RIGID INDENTOR OVER A POWER LAW VISCOELASTIC HALFSPACE	5. TYPE OF REPORT & PERIOD COVERED: Interim
	6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s): J. R. Walton, A. Nachman R. A. Schapery	8. CONTRACT OR GRANT NUMBER(s): AFOSR - 3290 - 77
9. PERFORMING ORGANIZATION NAME AND ADDRESS: Texas A&M University, Mathematics Department, College Station, TX 77843	10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS: 61102F 2304/A4
11. CONTROLLING OFFICE NAME AND ADDRESS: Air Force Office of Scientific Research/NM, Bolling AFB, Washington, DC 20332	12. REPORT DATE: 1977
	13. NUMBER OF PAGES: 52
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office): 56p.	15. SECURITY CLASS. (of this report): UNCLASSIFIED
	15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Power law viscoelastic material, generalized Abel equation, Riemann boundary value problems, sliding friction, dual integral equations

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

Closed form solutions are obtained for the problem of a rigid asperity sliding with Coulomb friction over a power law viscoelastic halfspace. The dual integral equations relating the unknown normal traction under the contact interval (also unknown) to the unknown normal displacement outside the contact interval are solved by first reducing the system to a generalized Abel integral equation and then appealing to the theory of Riemann-Hilbert boundary value problems. The physical quantities of

DD FORM 1473 1 JAN 73 EDITION OF 1 NOV 65 IS OBSOLETE

UNCLASSIFIED
SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

20. cont'd.

interest (eg. the coefficient of sliding friction) are determined for the three canonical indentors: a parabolic punch, a wedge punch and a flat punch. It is observed that for certain power law materials, singularities in the normal traction field occur even for the smooth parabolic indentor.

AFOSR-TR- 77-0995

The Sliding of a Rigid Indentor Over a Power Law Viscoelastic Halfspace

By

J. R. Walton[1] and A. Nachman[2]

Department of Mathematics
Texas A&M University
College Station, Texas 77843

and

R. A. Schapery
Departments of Civil and Aeronautical Engineering
Texas A&M University
College Station, Texas 77843

1,2. Supported in part by the United States Air Force under AFOSR Grant 77-3290.

Summary.

Closed form solutions are obtained for the problem of a rigid asperity sliding with Coulomb friction over a power law viscoelastic halfspace. The dual integral equations relating the unknown normal traction under the contact interval (also unknown) to the unknown normal displacement outside the contact interval are solved by first reducing the system to a generalized Abel integral equation and then appealing to the theory of Riemann-Hilbert boundary value problems. The physical quantities of interest (eg. the coefficient of sliding friction) are determined for the three canonical indentors: a parabolic punch, a wedge punch and a flat punch. It is observed that for certain power law materials, singularities in the normal traction field occur even for the smooth parabolic indentor.

Introduction.

Sliding and rolling contact problems in linear viscoelasticity theory have received considerable attention by analysts during the last two decades; for example, see [3], [4], [7] and the recent review by Rutanaprakarn [6]. The primary objective of these studies is to predict the dependence of the contact pressure distribution, friction coefficient (or mechanical dissipation) and contact area on the applied normal force, speed and viscoelastic properties of the slider (unless rigid) and the other body on which contact is made. It should be noted that without adhesion and surface roughness the sliding contact problem is equivalent to that of rolling contact for rigid circular cylinders and spheres pressed against deformable media.

In none of these studies is an exact analytical solution found, except the case in which the material is characterized as a so-called standard linear solid (SLS) [3]. This type of representation is far too simple to fit actual viscoelastic data. Indeed, for rubbery materials, the modulus varies with time or frequency over typically ten to twenty decades (e.g. [1], [7], [8]); at best, the SLS can be fit, approximately, to actual data over two decades.

One very important feature of actual viscoelastic behavior is that it often obeys a power law in time or frequency over much of its range of variation. On this basis, we believe that it would be very desirable to obtain analytical solutions to problems involving power law materials. The behavior predicted with such a model would certainly provide a better mathematical description of contact phenomena than with the SLS, and should be useful in guiding and checking numerical solutions to problems in which real viscoelastic data are used.

Inasmuch as a power law does not fit actual data over the entire range of variation, it would be helpful to know if significant error is introduced in contact solutions because of this lack of complete agreement. Rutanaprakarn [6] deals with this question, and one can infer from this study (especially Figs. 5 and 7 in [6]) that for a given sliding speed, any descrepancy is unimportant outside of a so-called "dominant frequency range". This range is limited to approximately four decades for power law exponents which do not exceed one-half; such is the case for practically all viscoelastic solids (see [1], [7], [8]). Also given in [6] is a procedure for selecting the power law exponent and coefficient from actual data when the latter values are not accurately characterized by a power law over any four-decade range.

In Section 1 we consider the problem of a rigid indentor sliding over a viscoelastic half-space and exhibit the governing equations. In that section the mixed boundary value problem is reduced to a set of dual integral equations. Section 2 contains an analysis of the dual integral equations based upon solving an equivalent Riemann-Hilbert boundary value problem. The results of Section 2 are applied in Section 3 to the contact problem for three canonical indentors: a flat punch, a parabolic punch and a wedge punch. Consistent with contact problems for a linearly elastic material, the appearance of corners produces singularities in the normal traction field under the indentor for our power law viscoelastic material. However, for certain power law materials we find jump discontinuities and singularities occurring even for smooth indentors. We conclude Section 3 with a brief discussion of complications attendent with multiple indentor problems for viscoelastic materials which are not encountered in elasticity.

Section 4 contains an interesting observation which suggests that the moving asperity problem may be cast in a variational framework. This is accomplished by demonstrating that for a parabolic indentor the contact interval, which is unknown apriori, may be determined by minimizing a suitable functional.

Section 1. Formulation of the Problem.

The particular problem which will occupy our attention in this investigation is the steady translation (to the left, say, with velocity U) of a rigid 2-dimensional asperity pressed against the 2-dimensional viscoelastic halfspace, $y \geq 0$.

Our ultimate goal is a description of the normal traction field on the contact interval and the vertical displacement of the halfspace outside the contact interval.

The mathematical formulation is as follows.

Neglecting inertia, the force balance equations are

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = 0$$

$$\frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} = 0$$

and the boundary conditions are

$$\tau_{xy}(x,o,t) = k\sigma_y(x,o,t) \qquad -\infty < x < \infty$$

$$v(x,o,t) = f(x + Ut) \qquad a(t) < x < b(t)$$

$$\sigma_y(x,o,t) = 0 \qquad x < a(t),\ x > b(t)$$

$$a(t) = a_o + Ut \qquad b(t) = b_o + Ut$$

$$\int_{a(t)}^{b(t)} \sigma_y(x,o,t)dx = \lambda = \text{total load.}$$

Here $v(x,y,t)$ and $u(x,y,t)$ are the vertical and horizontal displacements, σ_x is the normal stress on x = constant lines, σ_y is the normal stress on y = constant lines, τ_{xy} is the shear stress, k is a given nonnegative constant

and $f(x)$ is the shape of the indentor. Adopting the standard notation

$\varepsilon_x = \frac{\partial u}{\partial x}$, $\varepsilon_y = \frac{\partial v}{\partial y}$, $\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$, $\Delta = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}$ for the strains we write

the viscoelastic stress strain laws as

$$\sigma_x = \int_{-\infty}^{t} \Lambda(t-\tau)\frac{\partial \Delta}{\partial \tau}\, d\tau + 2 \int_{-\infty}^{t} G(t-\tau)\frac{\partial \varepsilon_x}{\partial \tau}\, d\tau$$

$$\sigma_y = \int_{-\infty}^{t} \Lambda(t-\tau)\frac{\partial \Delta}{\partial \tau}\, d\tau + 2 \int_{-\infty}^{t} G(t-\tau)\frac{\partial \varepsilon_y}{\partial \tau}\, d\tau$$

$$\tau_{xy} = \int_{-\infty}^{t} G(t-\tau)\frac{\partial \gamma_{xy}}{\partial \tau}\, d\tau.$$

To employ Fourier transforms it is necessary to differentiate all the equations (save for the equation $\sigma_y(x,o,t) = 0$, $x < a(t)$, $x > b(t)$) with respect to x and then adopt the Gallilean variable $s = x + Ut$.

Define the Fourier transform of any function $g(x,y)$ as

$$\overline{g(p,y)} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ips}\, g(s,y)\, ds.$$

Since $\Lambda(z)$ and $G(z)$ vanish for $z < 0$ it follows that

$$\frac{1}{\sqrt{2\pi}} \overline{\left(\frac{\partial \sigma_x}{\partial s}\right)} = ip\,\bar{\Lambda}\, \overline{\left(\frac{\partial \Delta}{\partial s}\right)} + 2ip\bar{G}\, \overline{\left(\frac{\partial \varepsilon_x}{\partial s}\right)}$$

$$\frac{1}{\sqrt{2\pi}} \overline{\left(\frac{\partial \sigma_y}{\partial s}\right)} = ip\,\bar{\Lambda}\, \overline{\left(\frac{\partial \Delta}{\partial s}\right)} + 2ip\bar{G}\, \overline{\left(\frac{\partial \varepsilon_y}{\partial s}\right)}$$

$$\frac{1}{\sqrt{2\pi}} \overline{\left(\frac{\partial \tau_{xy}}{\partial s}\right)} = ip\bar{G}\, \overline{\left(\frac{\partial \gamma_{xy}}{\partial s}\right)}$$

and

$$ip\, \overline{\left(\frac{\partial \sigma_x}{\partial s}\right)} + \frac{\partial}{\partial y} \overline{\left(\frac{\partial \tau_{xy}}{\partial s}\right)} = 0$$

$$\frac{\partial}{\partial y}\left(\overline{\frac{\partial\sigma_y}{\partial s}}\right) + ip\left(\overline{\frac{\partial\tau_{xy}}{\partial s}}\right) = 0$$

$$\left(\overline{\frac{\partial\varepsilon_x}{\partial s}}\right) = ip\left(\overline{\frac{\partial u}{\partial s}}\right), \quad \left(\overline{\frac{\partial\varepsilon_y}{\partial s}}\right) = \frac{\partial}{\partial y}\left(\overline{\frac{\partial v}{\partial s}}\right)$$

$$\left(\overline{\frac{\partial\gamma_{xy}}{\partial s}}\right) = \frac{\partial}{\partial y}\left(\overline{\frac{\partial u}{\partial s}}\right) + ip\left(\overline{\frac{\partial v}{\partial s}}\right), \quad \left(\overline{\frac{\partial\Delta}{\partial s}}\right) = \frac{\partial}{\partial y}\left(\overline{\frac{\partial v}{\partial s}}\right) + ip\left(\overline{\frac{\partial u}{\partial s}}\right).$$

Thus we arrive at

$$\overline{G}\,\frac{\partial^2}{\partial y^2}\left(\overline{\frac{\partial u}{\partial s}}\right) + ip(\overline{\Lambda} + \overline{G})\frac{\partial}{\partial y}\left(\overline{\frac{\partial v}{\partial s}}\right) - (\overline{\Lambda} + 2\overline{G})p^2\left(\overline{\frac{\partial u}{\partial s}}\right) = 0$$

$$(\overline{\Lambda} + 2\overline{G})\,\frac{\partial^2}{\partial y^2}\left(\overline{\frac{\partial v}{\partial s}}\right) + ip(\overline{\Lambda} + \overline{G})\frac{\partial}{\partial y}\left(\overline{\frac{\partial u}{\partial s}}\right) - p^2\overline{G}\left(\overline{\frac{\partial v}{\partial s}}\right) = 0$$

which yields

$$\left(\overline{\frac{\partial v}{\partial s}}\right) = A_1(p)e^{-|p|y} + A_2(p)ye^{-|p|y}$$

$$\left(\overline{\frac{\partial u}{\partial s}}\right) = -i\frac{p}{|p|}A_1(p)e^{-|p|y} - i\frac{p}{|p|}A_2(p)ye^{-|p|y} + \frac{i}{p}\,\frac{\overline{\Lambda}+3\overline{G}}{\overline{\Lambda}+\overline{G}}\,A_2(p)e^{-|p|y}.$$

The condition $\frac{\partial\overline{\tau}_{xy}}{\partial s}(s,o) = k\,\frac{\partial\overline{\sigma}_y}{\partial s}(s,o)$ results in $A_1(i + k|p|/p)p^2 = A_2[i|p|(\overline{\Lambda} + 2\overline{G}) + kp\overline{G}]/(\overline{\Lambda} + \overline{G})$.

Before proceeding we make the following explanatory observations. The elastic version of our stress-strain laws would be

$$\sigma_x = \Lambda\Delta + 2G\varepsilon_x, \text{ etc.}$$

where Λ and G are constants. If we take E to be Young's modulus and ν to be Poisson's ratio then it is well known that $\frac{G(\Lambda+G)}{\Lambda+2G} = \frac{E}{4(1-\nu^2)}$. Now if we

take $\tilde{G} = ip\bar{G}$ and $\tilde{\Lambda} = ip\bar{\Lambda}$ and appeal to standard correspondence principles (see [7]) it follows that

$$\frac{\tilde{G}(\tilde{\Lambda} + \tilde{G})}{\tilde{\Lambda} + 2\tilde{G}} = \frac{\tilde{E}}{4(1-\tilde{\nu}^2)} .$$

For a wide variety of materials $\tilde{\nu}^2$ is a constant (see [7]). Therefore it suffices to assume merely that $E(t) = \mathcal{E} t^{-\alpha} H(t)$ ($H(t)$ is the Heavyside function) to characterize our power law viscoelastic material. It should also be remembered that once $s = x + Ut$ is chosen as a new independent variable both Λ and G are functions of t/U essentially and hence

$$\bar{E} = \frac{U^{\alpha}\mathcal{E}\Gamma(1-\alpha)(ip)^{\alpha-1}}{\sqrt{2\pi}}$$

Since

$$\frac{\overline{\partial\sigma}}{\partial s}(s,o) = 2(2\pi)^{1/2}(ip)|p| \frac{\bar{G}(\bar{\Lambda} + \bar{G})}{(\bar{\Lambda} + 2\bar{G})}\left[1 - \frac{\bar{G}\, ik\, \mathrm{sgn}(p)}{(\bar{\Lambda} + 2\bar{G})}\right]^{-1} A_1(p),$$

$$\begin{aligned}\sigma_y(s,y) &= \mathcal{F}^{-1}\{\overline{\sigma_y}(p,y);s\} \\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ips}\,\overline{\sigma_y}(p,y)\,dp \\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ips}\,(ip)^{-1}\overline{\left(\frac{\partial\sigma_y}{\partial s}\right)}\,dp ,\end{aligned}$$

and

$$\frac{\bar{G}}{\bar{\Lambda} + 2\bar{G}} = \frac{\frac{1}{2} - \tilde{\nu}}{1 - \tilde{\nu}}$$

we obtain

$$\sigma_y(s,o) = \frac{U^{\alpha}\mathcal{E}\Gamma(1-\alpha)}{2\sqrt{2\pi}(1-\tilde{\nu}^2)}\int_{-\infty}^{\infty}\frac{e^{ips}|p|(ip)^{\alpha-1}A_1(p)\,dp}{\left(1 - ik\left(\frac{1/2 - \tilde{\nu}}{1 - \tilde{\nu}}\right)\mathrm{sgn}(p)\right)} .$$

By defining

(1) $$g(s) = 2(1-\tilde{\nu}^2)\sigma_y(s,o)/(U^\alpha \delta \Gamma(1-\alpha))$$

and

$$\beta = \frac{2}{\pi}\tan^{-1}\left[k\,\frac{1/2-\tilde{\nu}}{1-\tilde{\nu}}\right],$$

the boundary conditions for $v(s,o)$ and $\sigma_y(s,o)$ yield the following dual integral equations

(2) $$\mathcal{F}^{-1}\{A_1(p)\} = f'(s) \qquad a_o < s < b_o$$

(3) $$\mathcal{F}^{-1}\{A_1(p)|p|(ip)^{\alpha-1}/[1-\tan(\beta\pi/2)\,\mathrm{sgn}\,(p)];s\} = g(s)$$

where $g(s)$ vanishes for $s < a_o$ and $b_o < s$ and is unknown for $a_o < s < b_o$. It should be noted that the parameter β satisfies $0 \leq \beta < 1$ and that $\beta = 0$ corresponds to frictionless sliding. The total load condition requires $g(s)$ to be integrable on (a_o,b_o) so that Eq. (3) may be inverted to obtain

(4) $$A_1(p) = [1 - i\tan(\beta\pi/2)\mathrm{sgn}(p)]\,|p|^{-1}(ip)^{1-\alpha}(2\pi)^{-1/2}\int_{a_o}^{b_o} g(s)e^{isp}\,ds.$$

Substituting (4) into (2) yields

(5) $$f'(s) = \mathcal{F}^{-1}\{|p|^{-1}(ip)^{\alpha-1}\mathcal{F}\{g(y);p\};s\}$$
$$-i\tan(\beta\pi/2)\,\mathcal{F}^{-1}\{p^{-1}(ip)^{\alpha-1}\mathcal{F}\{g(y);p\};s\}.$$

The right side of (5) may be simplified as follows:

(6) $$\mathcal{F}^{-1}\{|p|^{-1}(ip)^{1-\alpha}\mathcal{F}\{g(y);p\};s\}$$
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}|p|^{-1}(ip)^{1-\alpha}e^{ips}dp\int_a^b g(y)e^{-iyp}dy$$

$$(6) \qquad = \frac{1}{2\pi}\int_{a_o}^{b_o} g(y)\,dy \int_{-\infty}^{\infty} |p|^{-1}(ip)^{1-\alpha} e^{ip(s-y)}\,dp$$

$$= \frac{\Gamma(1-\alpha)}{\pi}\left[\int_s^{b_o} \frac{g(t)dt}{(t-s)^{1-\alpha}} - \cos\alpha\pi \int_{a_o}^{b_o} \frac{g(t)dt}{(s-t)^{1-\alpha}}\right] .$$

The interchange required for line (6) is readily justified since g(t) is assumed integrable. In a similar way it may be shown that

$$(7) \qquad \mathcal{F}^{-1}\{p^{-1}(ip)^{1-\alpha}\mathcal{F}\{g(y);p\};s\}$$

$$= i\,\sin(\alpha\pi)\frac{\Gamma(1-\alpha)}{\pi}\int_{a_o}^{s} \frac{g(t)dt}{(s-t)^{1-\alpha}} .$$

Substituting (6) and (7) into (5) results in the following integral equation relating $f'(s)$ and $g(s)$:

$$(8) \qquad \frac{f'(s)\pi}{\Gamma(1-\alpha)} = \int_s^{b_o} \frac{g(t)dt}{(t-s)^{1-\alpha}} - \frac{\cos(\alpha+\beta/2)\pi}{\cos(\beta\pi/2)}\int_{a_o}^{s} \frac{g(t)dt}{(s-t)^{1-\alpha}} .$$

Equation (8) is a generalized Abel type integral equation. The special case $\alpha + \beta/2 = 1/2$ produces an elementary Abel equation which may be readily inverted to yield

$$g(t) = -\frac{\sin\alpha\pi}{\Gamma(1-\alpha)}\frac{d}{dt}\int_t^{b_o} \frac{f'(s)}{(s-t)^{\alpha}}ds .$$

However, obtaining a solution for (8) when $\alpha + \beta/2 \neq 1/2$ requires much more sophisticated techniques. One approach is to map the problem onto the interval $(0,\infty)$ and use Mellin transforms. Computing the resulting inverse transforms requires summing certain infinite series of residues.

The series obtained in this way are expressible in terms of hypergeometric functions.

A far more elegant closed form solution of (8) from which all physical quantities of interest can be computed may be obtained by solving a certain Riemann-Hilbert boundary value problem that is equivalent to (8). This technique has been described by Gakov [2] and is developed in the next section.

Section 2. Analysis of the Riemann Boundary Value Problem.

Define the function of a complex variable z analytic in the complex plane cut along the segment $[a_o,b_o]$ on the real axis by

$$\Phi(z) = [(z-a_o)(b_o-z)]^{-\alpha/2} \int_{a_o}^{b_o} \frac{g(t)}{(t-z)^{1-\alpha}}\, dt.$$

The multivalued function $[(z-a_o)(b_o-z)]^{-\alpha/2}(t-z)^{\alpha-1}$ is defined and analytic in a plane cut along $[a_o,b_o]$ by some branch. The density g(t) is assumed to have the representation

$$g(t) = \frac{g^*(t)}{[(t-a_o)(b_o-t)]^{\alpha}}$$

where $g^*(t)$ is Holder continuous on $[a_o,b_o]$. Define, for convenience,

$$R(z) = [(z-a_o)(b_o-z)]^{\alpha/2} .$$

Then $\Phi(z) = \frac{1}{R(z)} \int_{a_o}^{b_o} \frac{g(t)}{(t-z)^{1-\alpha}}\, dt$. Note that $\Phi(z)$ is analytic in the plane cut along $[a_o,b_o]$ and that

$$\Phi(z) = \frac{1}{z} \quad \text{as} \quad z \to \infty,$$

(9) $$\Phi(z) = 0[(z-a_o)^{-\alpha/2}] \text{ and}$$

$$\Phi(z) = 0[(b_o-z)^{-\alpha/2}] .$$

Let $\Phi^{\pm}(x)$ denote the limiting values of $\Phi(z)$ as z approaches the cut from above and below respectively. It is readily shown (see Gakov [2]) that on $[a_o,b_o]$

$$\Phi^{+}(x) = \frac{1}{R(x)}\left[-e^{-\alpha\pi i}\int_{a_o}^{x}\frac{g(t)dt}{(x-t)^{1-\alpha}} + \int_{x}^{b_o}\frac{g(t)dt}{(t-x)^{1-\alpha}}\right]$$

and

$$\Phi^{-}(x) = -\frac{1}{R(x)}\left[\int_{a_o}^{x}\frac{g(t)dt}{(x-t)^{1-\alpha}} - e^{-\alpha\pi i}\int_{x}^{b_o}\frac{g(t)dt}{(t-x)^{1-\alpha}}\right] .$$

From these two relations it is seen that

(10) $$\int_{a_o}^{x}\frac{g(t)dt}{(x-t)^{1-\alpha}} = \left[\frac{-e^{-\alpha\pi i}\Phi^{+}(x) + \Phi^{-}(x)}{e^{-2\alpha\pi i} - 1}\right] R(x)$$

and

(11) $$\int_{x}^{b_o}\frac{g(t)dt}{(t-x)^{1-\alpha}} = -\left[\frac{\Phi^{+}(x) - e^{-\alpha\pi i}\Phi^{-}(x)}{e^{-2\alpha\pi i} - 1}\right] R(x) .$$

Substituting (10) and (11) into (8) we obtain, finally, a Riemann boundary value problem for the determination of $\Phi(z)$. That is, it is required to find a function $\Phi(z)$ analytic in the complex plane cut along the segment $[a_o, b_o]$ subject to the asymptotic conditions (9) and whose limiting values $\Phi^{+}(x)$ and $\Phi^{-}(x)$ satisfy on the cut the equation

(12) $$\Phi^{+}(x) = G(x)\Phi^{-}(x) + g_1(x)$$

where $G(x) = -e^{i\pi(\alpha+\beta)}$ and $g_1(x) = \dfrac{2\pi f'(x)}{\Gamma(1-\alpha)R(x)(1-i\tan(\beta\pi/2))}$.

Once $\Phi(z)$ has been determined, $g(t)$ may be obtained by first computing the limits $\Phi^{\pm}(x)$ and then solving either of the Abel integral equations (10) or (11).

The index, χ, of the Riemann problem depends upon the class of functions

in which a solution is sought. In particular, if the solution is to be bounded at both endpoints, χ is -1; if it is allowed to be unbounded at both endpoints χ is $+1$; while if it is to be bounded at one endpoint and unbounded at the other χ is zero.

Define the auxiliary function $\Phi_1(z)$ by

$$\Phi^+(z) = (z-a_o)^\gamma (z-b_o)^{\gamma'} \Phi_1^+(z) \qquad \text{Im}(z) > 0$$

$$\Phi^-(z) = \left[\frac{z-a_o}{z-z_o}\right]^\gamma \left[\frac{z-b_o}{z-z_o}\right]^{\gamma'} \Phi_1^-(z) \qquad \text{Im}(z) < 0\ .$$

The branch cut for $\left[\frac{z-a_o}{z-z_o}\right]^\gamma$ consists of a line segment from z_o to a_o connected to the interval $[a_o,\infty)$ along the x-axis whereas for $\left[\frac{z-b_o}{z-z_o}\right]^{\gamma'}$ it consists of a line segment from z_o to b_o connected to the interval $[b_o,\infty)$. The exponent γ is chosen to be $-\left(\frac{\alpha+\beta-1}{2}\right)$ if $\Phi(z)$ is to be bounded at a_o and $-\left(\frac{\alpha+\beta+1}{2}\right)$ otherwise. The exponent γ' is then chosen to be $-(\gamma+\chi)$, where χ is the index. The function $\Phi_1(z)$ may be determined by solving the Riemann problem with boundary equation

$$\Phi_1^+(x) = (x-z_o)^\chi \Phi_1^-(x) + (x-a_o)^{-\gamma}(x-b_o)^{-\gamma'} g_1(x). \tag{13}$$

The equation (13) is assumed to hold on a semicircle in the upper half-plane with $[a_o,b_o]$ as diameter and z_o in the interior. If $\chi = 1$ the solution to (13) that is $0\left(\frac{1}{z}\right)$ as $z \to \infty$ is given by

$$\Phi_1(z) = X_1(z)\,\frac{1}{2\pi i}\left[\int_{a_o'}^{b_o} \frac{(\tau-a_o)^{-\gamma}(\tau-b_o)^{-\gamma'} g_1(\tau)\,d\tau}{\tau - z} + C\right] \tag{14}$$

where C is an arbitrary constant and the function $X_1(z)$, the canonical solution of the homogeneous problem corresponding to (13), is given by

$$X_1^+(z) = 1 \text{ and } X_1^-(z) = (z-z_o)^{-\chi}.$$

In (14) the integral is to be interpreted as a Cauchy principal value. If $\chi = 0$ or -1 then $C = 0$, and when $\chi = -1$ in order to obtain a solution it is necessary to impose the orthogonality condition

$$\int_{a_o}^{b_o} (\tau-a_o)^{-\gamma}(\tau-b_o)^{-\gamma'} g_1(\tau)d\tau = 0. \tag{15}$$

Before proceeding to a discussion of the form of the solution for the possible values of χ we shall consider the question of uniqueness of the solution for the dual integral equations. Since the problem is linear this is tantamount to assuming $f'(x) = 0$ for $a_o < x < b_o$, and hence that $g_1(x) = 0$. We shall allow only the physically reasonable solutions $A_1(t)$ which correspond to a finite total load, that is for which $g(x)$ is integrable. The homogeneous dual integral equations possess a non-zero solution only if the homogeneous Riemann problem corresponding to (13) possesses non-trivial solutions. From (14) non-trivial solutions exist if and only if $\chi = 1$ and are of the form

$$\Phi_1^+(z) = C, \quad \Phi_1^-(z) = (z-z_o)^{-1}C$$

where C is an arbitrary constant. The function $\Phi(z)$ is then given by

$$\Phi^+(z) = C(z-a_o)^{-(\alpha+\beta+1)/2}(z-b_o)^{(\alpha+\beta-1)/2}$$

$$\Phi^-(z) = -C(z-a_o)^{-(\alpha+\beta+1)/2}(z-b_o)^{(\alpha+\beta-1)/2}e^{-i\pi(\alpha+\beta)}.$$

The pnase shift in the expression for $\Phi^{-}(z)$ results from the two different branch cuts used for $[z-z_0]^{-\gamma}$ and $[z-z_0]^{-\gamma'}$. The limits $\Phi^{\pm}(x)$ are easily computed and when substituted into (11) yield the following Abel integral equations for g(x)

(16) $$\int_x^{b_o} \frac{g(t)dt}{(t-x)^{1-\alpha}} = C'(x-a_o)^{-(\beta+1)/2}(b_o-x)^{(\beta+2\alpha-1)/2} ,$$

(17) $$\int_{a_o}^{x} \frac{g(t)dt}{(x-t)^{1-\alpha}} = C''(x-a_o)^{-(\beta+1)/2}(b_o-x)^{(\beta+2\alpha-1)/2}$$

where C' and C" are constants.

From (17) we see that

$$\int_{a_o}^{x} g(t)dt = \frac{C''\sin\alpha\pi}{\pi} \int_{a_o}^{x} \frac{(t-a_o)^{-(\beta+1)/2}(b_o-t)^{(\beta+2\alpha-1)/2}dt}{(x-t)^{\alpha}} .$$

Observe that for x near a_o,

$$\int_{a_o}^{x} g(t)dt = O[(x-a_o)^{-(2\alpha+\beta-1)/2}]$$

from which it follows that

$$g(x) = O[(x-a_o)^{-(2\alpha+\beta+1)/2}] .$$

The function g(x) is integrable at a_o if and only if $0 < 2\alpha+\beta < 1$. Similarly, we obtain from (16) that for x near b_o

$$g(x) = O[(b_o-x)^{(\beta-1)/2}] .$$

The above analysis shows that the dual integral equations possess a unique solution in the class of functions for which $g(x)$ is integrable if and only if $2\alpha+\beta \geq 1$. Note, however, that the solution is unique for all values of $2\alpha+\beta$ if $g(x)$ is required to be square integrable.

In the subsequent sections we shall have need of the solutions of the generalized Abel equation (8) for $\chi = -1$ and $\chi = 0$. We list them here for future reference.

When $\chi = -1$ we obtain for $\Phi^{\pm}(z)$

$$\Phi^{+}(z) = \frac{(z-a_o)^{(1-\alpha-\beta)/2}(z-b_o)^{(\alpha+\beta+1)/2}e^{-\pi i(\alpha+\beta+1)/2}}{(i + \tan^{-1}(\beta\pi/2))\Gamma(1-\alpha)} \int_{a_o}^{b_o} \frac{(\tau-a_o)^{(\beta-1)/2}(b_o-\tau)^{-(\beta+2\alpha+1)/2}f'(\tau)d\tau}{(\tau - z)}$$

$$\Phi^{-}(z) = \frac{(z-a_o)^{(1-\alpha-\beta)/2}(z-b_o)^{(\alpha+\beta+1)/2}e^{-\pi i(\alpha+\beta+1)/2}}{(i+\tan^{-1}(\beta\pi/2))\Gamma(1-\alpha)e^{\pi i(\alpha+\beta+1)}} \int_{a_o}^{b_o} \frac{(\tau-a_o)^{(\beta-1)/2}(b_o-\tau)^{-(\beta+2\alpha+1)/2}f(\tau)d\tau}{(\tau - z)}.$$

The limits $\Phi^{+}(x)$ and $\Phi^{-}(x)$ are readily computed using the Plemelj formulas and when substituted into (10) and (11) yield the following Abel integral equations for the determination of $g(t)$:

$$(18) \quad \int_{a_o}^{x} \frac{g(t)dt}{(x-t)^{1-\alpha}} = \Gamma(\alpha)(1+[\tan^{-1}(\beta\pi/2)]^2)^{-1/2}[f'(x)\sin(\pi\beta/2)-\cos(\pi\beta/2)p(x)]$$

$$(19) \quad \int_{x}^{b_o} \frac{g(t)dt}{(t-x)^{1-\alpha}} = \Gamma(\alpha)(1+\tan^{-1}(\beta\pi/2)]^2)^{-1/2}[\sin[(\alpha+\beta/2)\pi]f'(x)-\cos[(\alpha+\beta/2)\pi]p(x)]$$

where

$$p(x) = (x-a_o)^{(1-\beta)/2}(b_o-x)^{(\beta+2\alpha+1)/2}\ \frac{1}{\pi}\int_{a_o}^{b_o} \frac{(\tau-a_o)^{(\beta-1)/2}(b_o-\tau)^{-(\beta+2\alpha+1)/2}f'(\tau)d\tau}{(\tau-x)}.$$

Of course, the above analysis is subject to the orthogonality condition (15) which now takes the form

$$\int_{a_o}^{b_o} (\tau-a_o)^{(\beta-1)/2}(b_o-\tau)^{-(\beta+2\alpha+1)/2}f'(\tau)d\tau = 0. \tag{20}$$

The calculations for $\chi = 0$ are similar and, for a solution that is bounded at a_o and unbounded at b_o, result in the following Abel equations

$$\int_{a_o}^{x} \frac{g(t)dt}{(x-t)^{1-\alpha}} = \Gamma(\alpha)\ \ (1+[\tan^{-1}(\beta\pi/2)]^2)^{-1/2}[f'(x)\sin(\pi\beta/2)-\cos(\pi\beta/2)p_1(x)] \tag{21}$$

$$\int_{x}^{b_o} \frac{g(t)dt}{(t-x)^{1-\alpha}} = \Gamma(\alpha)\ \ (1+\tan^{-1}(\beta\pi/2)]^2)^{-1/2}[\sin[(\alpha+\beta/2)\pi]f'(x)-\cos[(\alpha+\beta/2)\pi]p_1(x)] \tag{22}$$

where

$$p_1(x) = (x-a_o)^{(1-\beta)/2}(b_o-x)^{(\beta+2\alpha-1)/2}\ \frac{1}{\pi}\int_{a_o}^{b_o} \frac{(\tau-a_o)^{(\beta-1)/2}(b_o-\tau)^{-(\beta+2\alpha-1)/2}f'(\tau)d\tau}{(\tau-x)}.$$

In the next section we apply the results obtained above to an investigation of the behavior of the solutions to the moving asperity problem for three canonical indentor shapes.

Section 3. Examples.

In this section we shall discuss the behavior of the solution to the moving asperity problem for three canonical indentors, the flat punch, the wedge punch and the parabolic punch, since each one exhibits interesting features of the power law model. For the sake of simplicity we shall assume frictionless sliding, i,e. $\beta = 0$. Allowing β to be non-zero introduces no new behavior into the solutions and represents a straight forward extension of the techniques needed to treat the frictionless case.

In the physical problem the function $f'(x)$ is assumed to have a known functional form for a given loading and indentor shape on the unknown contact interval $[a_o, b_o]$. It is required to determine a_o, b_o and $\sigma_y(x,0)$, or equivalently, $g(x)$. Moreover, we shall also compute the coefficient of friction C_f given by

$$(23) \qquad C_f = \frac{1}{\lambda}\int_{a_o}^{b_o} f'(x)\sigma_y(x,0)\,dx$$

where λ is the total load given by

$$\lambda = \int_{a_o}^{b_o} \sigma_y(x,0)\,dx.$$

It is an interesting feature of viscoelastic material models that there is a resultant frictional like force impeding the frictionless sliding of a punch. It is generated by the bulging of the material ahead of the asperity.

We consider first the flat punch or square cornered block sliding over the material. This corresponds to $f'(x) = 0$ on the contact interval $[a_o, b_o]$. If the contact interval is to be non-degenerate, i.e. not just a singleton .

point, then $g(x)$ must be a non-trivial solution of the homogeneous generalized Abel integral equation (8). By the analysis of the last section it is apparent that an integrable function $g(x)$ satisfying the homogeneous equation over a non-degenerate interval will exist if and only if $0 < \alpha < 1/2$. Hence for $1/2 \leq \alpha < 1$ we are forced to conclude that $a_o = b_o$, that is, the material is in contact with the flat punch only at the leading edge and the problem becomes that of a moving point load! (See Figure 1.)

Insert fig. 1, 2

To analyze the flat punch problem for $1/2 < \alpha < 1$ it is simplest to appeal directly to the dual integral equations (2) and (3) for $\beta = 0$. Without loss of generality we may assume the contact point to be $a_o = 0$. Therefore, consider

$$\text{(24)} \qquad \mathcal{F}^{-1}\{A_1(p);s\} = f'(s) \qquad s \neq 0$$

$$\text{(25)} \qquad \mathcal{F}^{-1}\{A_1(p)|p|(ip)^{\alpha-1};s\} = c\delta(s)$$

where c is an arbitrary constant and $\delta(x)$ denotes the Dirac measure. From (25) we obtain

$$A_1(p) = c|p|^{-1}(ip)^{1-\alpha}$$

which when substituted into (24) yields

$$f'(s) = \frac{\Gamma(1-\alpha)c}{\pi} \begin{cases} -\cos\alpha\pi \; s^{\alpha-1} & s > 0 \\ |s|^{\alpha-1} & s > 0 \end{cases} .$$

The constant c is determined from knowing the total load. It is interesting to observe that when $\alpha = 1/2$, $f'(s) \equiv 0$ for $s > 0$ and that $f'(-0) = +\infty$. Hence, the displacement is constant (which may be taken to be zero) past

the contact point and has a vertical tangent as it approaches the contact point from the left. (See Figure 1.) For $1/2 < \alpha < 1$ the displacement to the left of the contact point is similar to that for $\alpha = 1/2$, whereas to the right of the contact point it is no longer constant. Rather, $f(s) = O(s^{\alpha})$ as s approaches positive infinity. (See Figure 1.)

The situation for $0 < \alpha < 1/2$ is completely different. Now the contact interval is the entire block face rather than merely one corner. From (16) we see that g(x) may be determined from

$$\int_{x}^{b_o} \frac{g(t)dt}{(t-x)^{1-\alpha}} = c \cot(\alpha\pi)(x-a_o)^{-1/2}(b_o-x)^{\alpha-1/2} .$$

Solving this Abel equation we obtain

$$g(t) = -\frac{c \cos \alpha\pi}{\pi} \frac{d}{dt} \int_{t}^{b_o} \frac{(x-a_o)^{-1/2}(b_o-x)^{\alpha-1/2}\, dx}{(x-t)^{\alpha}}$$

(26) $$= \frac{c\Gamma(1-\alpha)}{\sqrt{\pi}\,\Gamma(1/2-\alpha)} (b_o-a_o)^{\alpha}(b_o-t)^{-1/2}(t-a_o)^{-\alpha-1/2}.$$

The constant c is easily determined from the total load condition. (See Figure 2).

To investigate the displacement off the contact interval it is useful to notice that equation (8) is valid over the entire real axis. Hence we see that for $s > b_o$

(27) $$f'(s) = -\cos \alpha\pi \int_{a_o}^{b_o} \frac{g(t)dt}{(s-t)^{1-\alpha}} ,$$

whereas for $s < a_o$

$$f'(s) = \int_{a_o}^{b_o} \frac{g(t)dt}{(t-s)^{1-\alpha}} . \tag{28}$$

In particular

$$f(x) = \begin{cases} -0(s^{\alpha}) & s \to +\infty \\ -0((s-b_o)^{\alpha+1/2}, & s \to b_o+ \\ -0((a_o-s)^{1/2}) & s \to a_o- \\ -0(|s|^{\alpha}) & s \to -\infty . \end{cases}$$

Note that the displacement has a vertical tangent at both the leading and trailing edges of the punch. (See Figure 1.)

We now consider the case of a parabolic asperity, i.e. one for which $f'(s) = -s$. If $\alpha = 1/2$ the problem is trivial. We obtain for $g(t)$

$$g(t) = -\frac{1}{\sqrt{\pi}} \frac{d}{dt} \int_t^{b_o} \frac{f'(s)ds}{(s-t)^{1/2}} ds .$$

The function $g(t)$ will be continuous for $t = b_o$ only if $f'(b_o) = 0$, i.e. if $b_o = 0$. In that case we have

$$g(t) = \frac{2}{\sqrt{\pi}} \sqrt{-t} \qquad a_o < t < 0.$$

The point a_o is then easily determined from the load equation. However, we shall postpone this calculation until the general case has been considered.

When $0 < \alpha < 1/2$ we seek a solution to the Riemann problem that is bounded at both a_o and b_o. The index of the problem is then -1 and the function $g(x)$ may be obtained by solving either of the Abel equations (18) and (19) with $\beta = 0$. Before describing the solution $g(x)$ it is convenient to determine the endpoints of the contact interval $[a_o,b_o]$.

The orthogonality condition (20) becomes

$$\int_{a_o}^{b_o} (\tau-a_o)^{-1/2}(b_o-\tau)^{-\alpha-1/2}\,\tau d\tau = 0. \tag{29}$$

From (29) it is evident that $a_o < 0 < b_o$, i.e. the material wraps around the indentor. (See Figure 3.) The integral in (29) is easily evaluated and yields

$$L^{-\alpha}B(1/2,1/2-\alpha)[L\frac{(1/2-\alpha)}{(1-\alpha)} - b_o] = 0$$

where L denotes the length of the contact interval (b_o-a_o) and $B(\cdot,\cdot)$ is the beta function. We therefore obtain the relation

$$b_o = \frac{L(1/2-\alpha)}{(1-\alpha)} \quad . \tag{30}$$

The parameter L will be determined later from the load equation.

insert fig. 3,4

To determine g(x) it is convenient to first evaluate the Cauchy integral defining p(x). It is straight forward to show that

$$p(x) = -x\tan\alpha\pi - \Gamma(1/2)\Gamma(1/2-\alpha)(b_o-x)^{\alpha+1/2}$$

$$\cdot\{{}_2F_1(1/2,\alpha-1/2;\alpha+1/2;(b_o-x)/L\} - \frac{b_o}{L}\left(\frac{\alpha}{1/2+\alpha}\right)$$

$$\cdot\ {}_2F_1(1/2,\ \alpha+1/2;\alpha+3/2;(b_o-x)/L)\}.$$

The derivative of p(x) may now be taken and yields

$$p'(x) = -\tan\alpha\pi - \Gamma(1/2)\Gamma(1/2-\alpha)(b_o-x)^{\alpha-1/2}$$

$$\cdot\{[(x-a_o)/L]^{1/2}[b_o\alpha/L + (1/2-\alpha)] - {}_2F_1(1/2,\alpha-1/2;\alpha+1/2;(b_o-x)/L)\}$$

Since $\tan(\alpha\pi)f'(b_o-) = p(b_o-)$, the solution of (19) is given by

$$(32) \qquad g(t) = \frac{-\sin(\alpha\pi)\Gamma(\alpha)}{\pi} \int_t^{b_o} (x-t)^{-\alpha}[\sin\alpha\pi f''(x) - \cos\alpha\pi p'(x)]dx.$$

Using (31), (30) and the fact that $f''(x) = -1$, the integral defining $g(t)$ in (32) may be evaluated and after some manipulation yields

$$(33) \qquad g(t) = \frac{1}{\Gamma(1-\alpha)} \int_t^{b_o} (b_o-\tau)^{-1/2}(\tau-a_o)^{1/2-\alpha}d\tau$$

$$- \frac{\Gamma(1/2)(1/2-\alpha)L^{1-\alpha}}{\Gamma(1-\alpha)\Gamma(2-\alpha)\Gamma(\alpha+1/2)} \int_t^{b_o} \frac{(\tau-a_o)^{-1/2}(b_o-\tau)^{\alpha-1/2}d\tau}{(\tau-t)^{\alpha}} .$$

(See Figure 4.)

Now assume $1/2 < \alpha < 1$. For this case it is necessary to seek a solution of the Riemann problem that is bounded at only one endpoint since the orthogonality condition (29) is satisfied only for a degenerate interval $a_o = b_o$ and allowing this requires the material to pass through the asperity. It is easy to see that we must require the solution to be bounded at a_o and hence unbounded at b_o. The function $g(x)$ may therefore be determined from either of equations (21) and (22) with $\beta = 0$. Evaluating $p_1'(x)$ we obtain

$$p_1'(x) = -\tan\alpha\pi \quad + \frac{\Gamma(1/2)\Gamma(3/2-\alpha)}{\pi\Gamma(1-\alpha)}(b_o-x)^{\alpha-3/2}L^{1/2-\alpha}$$

$$\cdot\left\{\left[\frac{(\alpha-3/2)}{(1-\alpha)}L + b_o\right]\left[(x-a_o)/L\right]^{-1/2} + \frac{L}{(1-\alpha)}\, {}_2F_1(1/2,\alpha-3/2;\alpha-1/2;(b_o-x)/L)\right\} .$$

Since for $1/2 < \alpha < 1$ the orthogonality equation (29) does not hold, some other condition must be imposed to specify a_o and b_o. It is natural to choose the points a_o and b_o so as to enhance the endpoint behavior of $g(x)$. Choosing

$$(34) \qquad b_o = \left(\frac{3/2-\alpha}{1-\alpha}\right) L$$

forces $g(x)$ to be continuous at $x = a_o$ while, as with the case $0 < \alpha < 1/2$, choosing

$$b_o = \left(\frac{1/2-\alpha}{1-\alpha}\right) L$$

forces continuity at b_o. However, the condition (34) is physically impossible since it predicts, among other things, a total load of the wrong sign. The function $g(x)$ must be positive. From equation (8), it is clear that if $1/2 < \alpha < 1$ then $f'(s)$ must be positive on $[a_o,b_o]$. Since $f'(s) = -s$ this means that $b_o < 0$, whereas line (34) results in $b_o > 0$. Hence, we conclude that the condition (30) obtained for $0 < \alpha < 1/2$ is valid also for $1/2 \leq \alpha < 1$. Note, then that for $1/2 < \alpha < 1$ the material leaves the indentor before the apex. (See Figure 3.)

Given relation (30) it is straight forward to show that the formula (33) obtained for $g(t)$ when $0 < \alpha < 1/2$ is valid for $1/2 < \alpha < 1$. It should be observed that

$$g(s)\ \begin{cases} O([b_o - s]^{1/2}) & \text{for } s \to b_o^- \\ O([s-a_o]^{1/2-\alpha}) & \text{for } s \to a_o^+ \end{cases} .$$

Therefore, the stress is continuous at b_o for all $0 < \alpha < 1$; whereas if $\alpha = 1/2$ it has a jump discontinuity at $s = a_o$, if $0 < \alpha < 1/2$ it is continuous at $s = a_o$ and if $1/2 < \alpha < 1$ it has a singularity of order $\alpha - 1/2$. (See Figure 4.)

In order to determine L, and hence a_o and b_o, we apply the total load condition

$$\lambda = \int_{a_o}^{b_o} \sigma_y(s,o)ds$$

$$= \frac{U^\alpha \mathcal{E}\,\Gamma(1-\alpha)}{2(1-\tilde{\nu}^2)} \int_{a_o}^{b_o} g(s)ds.$$

For g(s) given by (33), this integral is readily computed and yields

$$\lambda = \frac{U^\alpha \mathcal{E}\Gamma(3/2)\Gamma(3/2-\alpha)L^{2-\alpha}}{2(1-\alpha)\Gamma(3-\alpha)(1-\tilde{\nu}^2)}$$

from which it follows that

$$L = \left[\frac{2\lambda(1-\alpha)\Gamma(3-\alpha)(1-\tilde{\nu}^2)}{U^\alpha \mathcal{E}\,\Gamma(3/2)\Gamma(3/2-\alpha)}\right]^{1/(2-\alpha)} .$$

It is useful to examine the stress and displacement profiles. We consider first the stress $\sigma_y(s,o)$. The derivative of g(s), given by (33), may be calculated easily and gives

$$g'(s) = \frac{(b_o-s)^{-1/2}(s-a_o)^{-\alpha-1/2}}{\Gamma(1-\alpha)}\left[\left(\frac{-\alpha}{1-\alpha}\right)L - s\right] .$$

Hence it is apparent that when $\alpha \geq 1/2$ $g(s)$ is monotone decreasing on (a_o, b_o). However, when $0 < \alpha < 1/2$, $g(s)$ has a unique maximum occurring at the point

$$s = \left(\frac{-\alpha}{1-\alpha}\right) L.$$

(See Figure 4.)

The behavior of the displacement profiles may be deduced from equations (27) and (28). It should be noted that the displacement has a continuously varying tangent over the whole line.

Mention should be given to the question of uniqueness for the parabolic indentor. As was demonstrated in the last section, when $1/2 \leq \alpha < 1$ uniqueness is assured, whereas, when $0 < \alpha < 1/2$ the solution to (8) is non-unique in the class of integrable functions. However, only one of the solutions is continuous. Adding to (33) any non-zero solution of the homogeneous problem results in a displacement profile whose tangent becomes vertical upon approaching the contact interval. This is obviously physically impossible since it corresponds to the material passing into the indentor.

As was indicated earlier, a quantity important in predicting the response of a viscoelastic material to loading is the friction coefficient given by (23). For the parabolic asperity C_f is easily computed. In particular if $f'(x) = -x$ on (a_o, b_o) then

$$C_f = \frac{\alpha(2-\alpha)}{(1-\alpha)(3-\alpha)} L .$$

For completeness we shall indicate what the solution for the parabolic asperity looks like when the friction term is included. For $g(t)$ we obtain

$$g(t) = \frac{1}{\Gamma(1-\alpha)} \int_t^{b_o} (b_o-x)^{(\beta-1)/2}(x-a_o)^{(1-\beta-2\alpha)/2}dx$$

$$- \frac{\Gamma((\beta+1)/2)(1-2\alpha-\beta)L^{1-\alpha}}{2\Gamma(1-\alpha)\Gamma(2-\alpha)\Gamma((1+\beta+2\alpha)/2)} \int_t^{b_o} \frac{(b_o-x)^{(\beta+2\alpha-1)/2}(x-a_o)^{-(\beta+1)/2}}{(x-t)^{\alpha}} dx.$$

The trailing contact point becomes

$$b_o = L/2 \ \frac{(1-2\alpha-\beta)}{(1-\alpha)}$$

where L comes from the load condition

$$\lambda = \frac{U^{\alpha}\mathcal{G}\Gamma((3-\beta-2\alpha)/2)\Gamma((\beta+3)/2)L^{2-\alpha}}{2(1-\tilde{\nu}^2)(1-\alpha)\Gamma(3-\alpha)} .$$

Finally for C_f we obtain

$$C_f = \frac{(\alpha+\beta)(2-\alpha)}{(3-\alpha)(1-\alpha)} L.$$

It should be noted that the parameter that governs the essential behavior of the solution in this case is $\alpha + \beta/2$ rather than α.

We next consider a wedge shaped punch for which

$$f'(x) = \begin{cases} S_1 & x < 0 \\ -S_2 & x > 0 \end{cases}$$

where S_1 and S_2 are positive constants. Assume first that $\alpha \geq 1/2$. Since $g(t)$ must be non-negative, we see from (8) that a_o and b_o must be such that $f'(x) \geq 0$ on (a_o,b_o). Hence $b_o \leq 0$.

For $\alpha = 1/2$ the problem is again trivial. The function $g(x)$ is given by

$$g(x) = \frac{S_1}{\sqrt{\pi}} (b_o - x)^{-1/2}.$$

(See Figure 6.) Recall that

$$f'(x) = \begin{cases} 0 & t > b_o \\ S_1 & a_o < t < b_o \\ \dfrac{S_1}{\pi} \displaystyle\int_{a_o}^{b_o} (b_o - t)^{-1/2} (t-x)^{-1/2} dt & t < a_o \end{cases}.$$

Obviously, from physical considerations we must take $b_o = 0$. The leading contact point is now easily determined from the load equation. (See Figure 5.) It should be noted that the stress is singular at the trailing contact point (the apex of the wedge) and has a jump discontinuity at the leading point. (See Figure 6.)

Insert
ig. 5, 6

When $1/2 < \alpha < 1$ we seek a solution to the Riemann problem that is bounded at a_o and unbounded at b_o. As before we solve the Abel equation (22) for $g(t)$ where $\beta = 0$ and $p_1(s)$ is given by

$$p_1(s) = \delta \tan \alpha\pi - \frac{\delta\Gamma(1/2)\Gamma(1/2-\alpha)(b_o - s)^{\alpha-1/2} L^{1/2-\alpha}}{\pi\Gamma(1-\alpha)}$$
$$\cdot {}_2F_1(1/2, \alpha - 1/2; \alpha + 1/2; (b_o - s)/L).$$

This reduces to solving

$$(35) \qquad \int_s^{b_o} \frac{g(t)dt}{(t-s)^{1-\alpha}} = h(s)$$

where

(36) $$h(s) = \frac{S_1\Gamma(\alpha)\Gamma(1/2)(b_o-s)^{\alpha-1/2}L^{1/2-\alpha}}{\Gamma(1-\alpha)\Gamma(1/2+\alpha)}\ {}_2F_1(1/2,\alpha-1/2;\alpha+1/2;(b_o-s)/L).$$

The derivative of h(s) is easily taken and yields

$$h'(s) = -\frac{S_1\Gamma(\alpha)\Gamma(1/2)L^{1-\alpha}(b_o-s)^{\alpha-3/2}(s-a_o)^{-1/2}}{\Gamma(1-\alpha)\Gamma(\alpha-1/2)}.$$

Solving for g(t) we obtain

$$g(t) = -\frac{\sin\alpha\pi}{\pi}\int_t^{b_o}\frac{h'(s)ds}{(t-s)^\alpha}$$

$$= \frac{S_1\Gamma(\alpha)\sin(\alpha\pi)\Gamma(1/2)}{\pi\Gamma(1-\alpha)\Gamma(\alpha-1/2)}L^{1-\alpha}\int_t^{b_o}\frac{(b_o-s)^{\alpha-3/2}(s-a_o)^{-1/2}}{(s-t)^\alpha}ds.$$

(See Figure 6.) It is easily seen that again we must choose $b_o = 0$. Indeed, recall that

(37) $$f'(s) = -\cos\alpha\pi\,\frac{\Gamma(1-\alpha)}{\pi}\int_{a_o}^{b_o}\frac{g(t)dt}{(s-t)^{1-\alpha}} \qquad s > b_o.$$

For $s > b_o$ $f'(s)$ is a decreasing function. Since $f'(s) = S_1$ for $a_o < s < b_o$ we conclude that $f'(s) < S_1$ when $s > b_o$. This is possible only if $b_o = 0$. Therefore, when $1/2 < \alpha < 1$ the material leaves the wedge indentor at the apex in contrast to the parabolic indentor for which it leaves in front of the peak. (See Figure 5.)

The load equation is readily computed and yields

$$\lambda = \frac{U^\alpha\delta\Gamma(3/2-\alpha)}{2\pi\Gamma(2-\alpha)(1-\nu^2)}L^{1+\alpha}$$

from which L, and hence a_o, may be determined. Note also that for $1/2 \leq \alpha < 1$ the friction coefficient, C_f, is just the slope of the leading face of the wedge, i.e. $C_f = S_1$.

It should be observed also that the stress is singular at both a_o and b_o. In particular

$$g(x) = \begin{cases} O((s-a_o)^{1/2-\alpha}) & s \to a_o+ \\ O((b_o-s)^{-1/2}) & s \to b_o- \end{cases} .$$

(See Figure 6.)

Assume now that $0 < \alpha < 1/2$. We may observe first that in this case $b_o > 0 > a_o$, i.e. the material must wrap around the indentor. (See Figure 5.) Indeed, from (37) we conclude that for $0 < \alpha < 1/2$, $f'(s)$ is nonnegative when $s > b_o$, from which it follows that $b_o \geq 0$. Moreover, if $b_o = 0$ then we may appeal to the analysis for the case $1/2 < \alpha < 1$ to conclude that $g(t)$ must solve (35) with $h(s)$ given by (36). Inverting (35) yields

$$g(t) = -\frac{\sin \alpha\pi}{\pi}\frac{d}{dt}\int_t^{b_o}\frac{h(s)dt}{(s-t)^{\alpha}} .$$

Observe that $h(s) = O[(b_o-s)^{\alpha-1/2}]$ for $s \to b_o-$ which implies that $g(t) = O[(b_o-t)^{-1/2}]$ for $s \to b_o-$. This, together with (37) shows that $f'(s) = O[(s-b_o)^{\alpha-1/2}]$ for $s \to b_o+$, from which it follows that the displacement possesses a vertical tangent upon approaching b_o. The last statement is a physical impossibility. Hence, $b_o > 0$.

To solve for $g(t)$ we seek a solution of the Riemann problem that is bounded at a_o and b_o. It is required, then, to solve equation (19) subject to the orthogonality constraint (20). However, for the wedge punch with $a_o < x < b_o$, the solution of equation (19) is not expressible

in terms of elementary functions but may be estimated numerically. Moreover, simple expressions for a_o and b_o are not obtainable, though it is possible to calculate a_o and b_o numerically as follows. Define a new parameter σ to be $\sigma = -a_o/L$. The orthogonality relation (20) becomes

$$0 = S_1 \int_o^{\sigma} z^{-1/2}(1-z)^{-\alpha - 1/2}dz - S_2 \int_{\sigma}^{1} z^{-1/2}(1-z)^{-\alpha - 1/2}dz. \tag{38}$$

The load equation is

$$\lambda = \frac{U^{\alpha} G\, \Gamma(1-\alpha)}{2(1-\nu^2)} \int_{a_o}^{b_o} g(s)ds \quad . \tag{39}$$

From (18) with $\beta = 0$ we obtain

$$\int_{a_o}^{x} \frac{g(t)dt}{(x-t)^{1-\alpha}} = -\Gamma(\alpha)p(x)$$

where p(t) is given by

$$p(t) = (t-a_o)^{1/2}(b_o-t)^{\alpha + 1/2} \frac{1}{\pi} \int_{a_o}^{b_o} \frac{(\tau-a_o)^{-1/2}(b_o-\tau)^{-\alpha - 1/2}}{(\tau - t)} f'(\tau)d\tau \; .$$

Hence,

$$\int_{a_o}^{b_o} g(t)dt = -\frac{1}{\Gamma(1-\alpha)} \int_{a_o}^{b_o} \frac{p(t)dt}{(b_o-t)^{\alpha}}$$

$$= \frac{1}{\pi\Gamma(1-\alpha)} \int_{a_o}^{b_o} f'(\tau)(\tau-a_o)^{-1/2}(b_o-\tau)^{-\alpha - 1/2} d\tau \int_{a_o}^{b_o} \frac{(t-a_o)^{1/2}(b_o-t)^{1/2}}{(t-\tau)} dt$$

$$= -\frac{1}{\Gamma(1-\alpha)} \int_{a_o}^{b_o} \tau(\tau-a_o)^{-1/2}(b_o-\tau)^{-\alpha - 1/2} d\tau$$

$$(40) \quad = -\frac{L^{1-\alpha}}{\Gamma(1-\alpha)} \left[S_1 \int_0^{\sigma} z^{1/2}(1-z)^{-\alpha - 1/2} dz - S_2 \int_{\sigma}^{1} z^{1/2}(1-z)^{-\alpha - 1/2} dz\right].$$

Line 40 follows from an obvious change of variables and repeated applications of (38). To determine a_o and b_o, one need only solve (38) numerically for σ and then use (39) and (40) to obtain L.

As was indicated above, it is not possible to express g(t) in terms of elementary functions. Once a_o and b_o are known g(t) may be approximated numerically. However, due to space limitations we shall not address this question here.

We remark at this time that our analysis yields as a special case, namely $\alpha = 0$, the solution to the problem of an indentor pressed into an elastic half space. Indeed, setting $\alpha = \beta = 0$ in the above expressions for g(t) produces the results obtained by Muskhelishvili [5] for the elastic contact problem. It should be noted that, as expected, the friction coefficient, C_f, vanishes when $\alpha = 0$ and the pressure distribution is symmetric on the contact interval for a symmetric punch.

As another remark we include a brief discussion of the problem of multiple asperities. In particular, assume that several indentors are pressed into the half-space and are moving with the same constant velocity. Assume that the indentors have shapes given by $f_1(x),\ldots,f_n(x)$ and that the

total pressure under each of the punches is known. Hence, we assume as known the n parameters

$$\lambda_i = \int_{a_i}^{b_i} \sigma_y(x,o)dx \qquad i = 1,\ldots,n$$

where $I_i = [a_i,b_i]$ is the unknown contact interval for the i^{th} indentor. Letting $g_i(t)$ denote the restriction of $g(t)$ (given by (1)) to the interval I_i, it is straight forward to show that $f_i(x)$ and $g_i(x)$ $i = 1,\ldots,n$ are related by the system of dual integral equations

$$\mathcal{F}^{-1}\{A_1(t);x\} = f_i(x) \qquad x \in I_i,\ i = 1,\ldots,n$$

$$\mathcal{F}^{-1}\{A_1(t)|t|(it)^{\alpha-1};x] = \sum_{i=1}^{n} \chi_{I_i}(x)g_i(x)$$

where

$$\chi_{I_i}(x) = \begin{cases} 1 & x \in I_i \\ 0 & x \notin I_i \end{cases} .$$

For simplicity, assume $n = 2$. Analogous to (8) we obtain the coupled system of integral equations for $g_i(t)$, $i = 1,2$

$$\frac{\pi f_1'(x)}{\Gamma(1-\alpha)} = \int_x^{b_1} \frac{g_1(t)dt}{(t-x)^{1-\alpha}} - \cos\alpha\pi \int_{a_1}^{x} \frac{g_1(t)dt}{(x-t)^{1-\alpha}} + \int_{a_2}^{b_2} \frac{g_2(t)dt}{(t-x)^{1-\alpha}} \qquad x \in I_1$$

(41)

$$\frac{\pi f_2'(x)}{\Gamma(1-\alpha)} = \cos\alpha\pi \int_{a_1}^{b_1} \frac{g_1(t)dt}{(x-t)^{1-\alpha}} + \int_x^{b_2} \frac{g_2(t)dt}{(t-x)^{1-\alpha}} - \cos\alpha\pi \int_{a_1}^{x} \frac{g_1(t)dt}{(x-t)^{1-\alpha}} \qquad x \in I_2.$$

The elastic contact problem for multiple indentors is no more difficult to solve than for a single punch. The situation for the viscoelastic half-space is considerably different. For the elastic half-space it is possible to reduce the problem to that of solving a single Riemann boundary value problem. Because of the occurence of branch points on each of the intervals I_i, no such reduction is possible for the viscoelastic problem. Rather one obtains a coupled system of Riemann boundary value problems for which no simple uncoupling seems to exist. In general, the system (41) must be solved numerically.

There is, however, one special case for which (41) is easily inverted. In particular, when $\alpha = 1/2$ the second equation uncouples and is easily inverted to determine $g_2(t)$. With $g_2(t)$ known, the first equation may be solved for $g_1(t)$. The four parameters a_i, b_i, $i = 1, 2$, are still to be deduced. The contact points a_2, b_2 for the trailing indentor are determined independently of the first indentor since the first indentor produces no effect on the second. The obvious coupling in (41a) is reflected in the choice of a_1 and b_1 and the resulting relative displacement of the two punches. For example, if $f_1'(x) = -x$ and $f_2'(x) = -(x-c)$ for $0 < c$ then $b_2 = c$,

$$a_2 = c - \left[\frac{\lambda_2(1-\tilde{\nu}^2)3\sqrt{\pi}}{2U^{1/2}\,\mathcal{E}}\right]^{2/3}$$

and a_1 and b_1 may be calculated from the equations

$$b_1 = -\frac{2}{\pi}\int_{a_2}^{c} (c-t)^{1/2}(t-b_1)^{-1/2}dt$$

$$\lambda_1 = \frac{U^{1/2}\mathcal{E}\sqrt{\pi}}{2(1-\nu^2)} \int_{a_1}^{b_1} g_1(t)\,dt$$

where

$$g_1(x) = \frac{2}{\pi}(b_1-x)^{1/2} - \frac{1}{\pi}\int_x^{b_1} \frac{dy}{(y-x)^{1/2}} \int_{a_2}^{c} (c-t)^{1/2}(t-y)^{-3/2}dt$$

and

$$g_2(x) = \frac{2}{\pi}(c-x)^{1/2}.$$

If the coordinate system is chosen so that $f(x) = -\frac{x^2}{2}$ $x \varepsilon I$, then on I_2 $f(x) = d - (c-x)^2/2$ where the relative displacement d is given by

$$d = (b_1-a_1)^2 - 2\int_{a_2}^{c}(c-z)^{1/2}(z-b_1)^{-1/2}dz$$

$$-4\int_{a_2}^{c}(c-z)^{1/2}(z-b_1)^{1/2}dz + (4 + 4/\pi)\int_{a_2}^{c}(c-z)^{1/2}(z-a_1)^{1/2}dz.$$

Section 4. Observations on determining the contact interval.

In the last section the criteria for determining the contact interval for each of the examples depended upon the particular indentor shape considered. No general principle other than preserving the physical integrity of the asperity was invoked. For more general indentors it may be very difficult to apply this criterion due to the complicated nature of the integrals involved.

In attempting to circumvent this difficulty, we conjectured that for a fixed loading, λ, the correct choice for a_o and b_o should correspond to a minimum, or at least stationary, point of C_f when considered as a function of a_o and b_o. In particular, we tested the conjecture by considering the following constrained minimization problem in the case of a parabolic indentor without friction:

$$\text{(42)} \qquad \text{minimize } \gamma C_f(L,a_o) = \int_{a_o}^{b_o} g(s)f'(s)ds \text{ subject to } \int_{a_o}^{b_o} g(s)ds = \gamma$$

where g(s) is given as the formal solution of (18) for $0 < \alpha < 1/2$ and of (21) for $1/2 \leq \alpha < 1$ and $\gamma = \dfrac{2\lambda(1-\tilde{\nu}^2)}{U^\alpha \mathcal{G}\,\Gamma(1-\alpha)}$.

The problem (42) may be solved by a standard Lagrange multiplier argument. We present the analysis for the case $1/2 < \alpha < 1$; the other cases may be treated similarly.

It may be shown that for $f'(s) = -s$

$$T(L,a_o) = \int_{a_o}^{b_o} g(s)(-s)ds \quad ,$$

$$= \frac{-1}{\Gamma(1-\alpha)} \left\{ - b_o \int_{a_o}^{b_o} \frac{p_1(s)ds}{(b_o-s)^\alpha} + \frac{1}{1-\alpha} \int_{a_o}^{b_o} p_1(s)(b_o-s)^{1-\alpha} ds \right\}$$

$$(43) = - \frac{1}{\Gamma(1-\alpha)} \Big\{ (L+a_o)[B(3/2,3/2 - \alpha)L^{2-\alpha} + a_o B(1/2,3/2 - \alpha)L^{1-\alpha}]$$

$$- \frac{1}{2(1-\alpha)}[B(5/2,3/2 - \alpha)L^{3-\alpha} - B(3/2,5/2 - \alpha)L^{3-\alpha}$$

$$+ a_o B(3/2,3/2 - \alpha)L^{2-\alpha} - a_o B(1/2,5/2 - \alpha)L^{2-\alpha}] \Big\} .$$

Similarly,

$$(44) \qquad \int_{a_o}^{b_o} g(s)ds = \frac{1}{\Gamma(1-\alpha)}[B(3/2,3/2 - \alpha)L^{2-\alpha} + a_o B(1/2,3/2 - \alpha)L^{1-\alpha}].$$

Hence we consider the problem of minimizing $T(L,a_o)$ subject to $G(L,a_o) = 0$ where $G(L,a_o) = \gamma - \int_{a_o}^{b_o} g(s)ds$. Using Lagrange multipliers, we consider the function

$$F(L,a_o,\xi) = T(L,a_o) + \xi G(L,a_o).$$

The partial derivatives $\frac{\partial F}{\partial L}$, $\frac{\partial F}{\partial a_o}$ and $\frac{\partial F}{\partial \xi}$ are easily computed from (43) and (44). It may be shown that the resultant variational equations possess a unique solution for which

$$L^{2-\alpha} = \frac{\gamma\Gamma(2-\alpha)\Gamma(3-\alpha)}{\Gamma(3/2)\Gamma(3/2 - \alpha)}$$

and

$$a_o = \frac{-1/2}{1 - \alpha} L .$$

This is the solution obtained in Section 3 and is the unique stationary

point of C_f.

For constant sliding velocity U and load λ, the coefficient of friction, C_f, is proportional to entropy production. Hence the above technique would be justified theoretically by proving a minimum entropy production principle. This topic will be considered in a subsequent paper. In this way it is possible to cast the moving asperity problem in a variational framework which provides a useful numerical approach for obtaining all the relevant physical quantities for general asperities.

Conclusion.

Morland [4] has shown that for the relatively general case of a Kelvin material the contact pressure distribution over a smooth indentor vanishes at the leading and trailing edges of the contact area, just as it does for elastic media. In contrast, however, we have found that for the parabolic asperity the pressure at the leading edge has a finite discontinuity when $\alpha = 1/2$ and an infinite discontinuity when $1/2 < \alpha < 1$. This behavior is apparently closely related to the fact that the initial value of the power law relaxation modules is infinite whereas the initial value of the Kelvin modulus is finite.

Nevertheless, the utility of our results, aside from the fact that closed form solutions are provided, is reflected by the formula for the friction coefficient. This coefficient, like the stress intensity factor for crack problems, has proved to be a useful characterization of the response of viscoelastic material to sliding contacts. When one recalls that linear elasticity theory predicts a physically unrealistic singular stress field at a crack tip but a finite, acceptably accurate intensity factor the analogy becomes even more appropriate.

The other interesting deviation from classical behavior that should not go unmentioned is the algebraic displacement to $\pm\infty$ of the material behind (ahead of) the asperity when $\alpha > 1/2$. This is to be contrasted to the two-sided logarithmic displacement to $-\infty$ for an elastic half space.

Bibliography

[1] J. D. Ferry, Viscoelastic behavior and analysis of composite materials, 2nd ed., John Wiley and Sons, New York, 1970.

[2] F. D. Gakhov, Boundary value problems, Pergamon Press, London, 1966.

[3] S. C. Hunter, "The rolling contact of a rigid cylinder with a viscoelastic halfspace," J. of Appl. Mech., 28, 611-617 (1961).

[4] L. W. Morland, "Exact solutions for rolling contact between viscoelastic cylinders," Quart. J. Mech. and Appl. Math., 20, 73-106 (1967).

[5] N. I. Muskhelishvili, Some basic problems of the mathematical theory of elasticity, 3rd ed., P. Noordhoff, Groningen, Holland, 1953.

[6] O. Rutanaprakarn, "Approximate solution for a rolling contact problem over a relatively general viscoelastic half-space," Ph.D. Dissertation, Texas A&M University, May 1977.

[7] R. A. Schapery, "Analysis of rubber friction: theoretical development with application of a numerical solution method based on the fast-Fourier transform," Texas A&M Univ. Rpt. MM 3043-76-2, March 1976.

[8] R. A. Schapery, "Viscoelastic behavior and analysis of composite materials," in Composite materials, Vol. 2, G. P. Sendeckyj, Ed., Academic Press, New York, 1974.

[9] M. L. Williams, "Structural analysis of viscoelastic materials," AIAA J., 785-808, 1964.

Figure 1. Displacement fields for a flat punch for $\alpha < 1/2$(———); $\alpha = 1/2$(-·-·-·); $\alpha > 1/2$(----).

Fig. 1

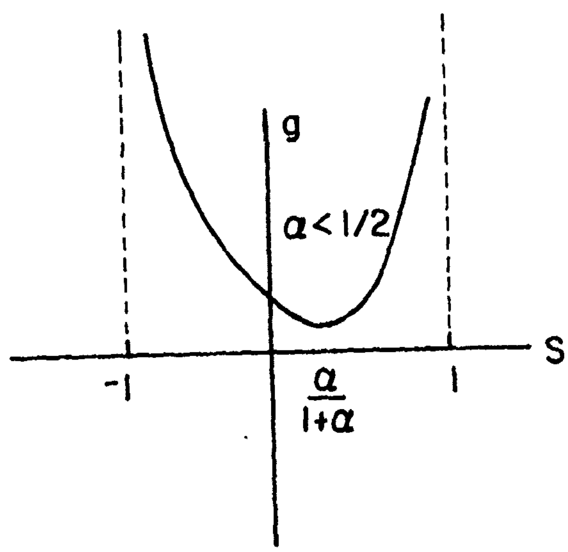

Figure 2. Normal traction field over the contact interval for a flat punch: $\alpha < 1/2$.

g. 2

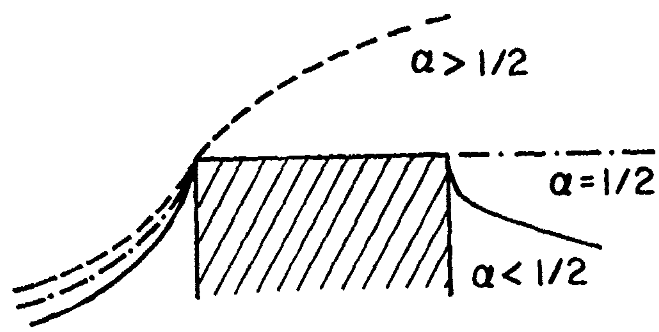

Figure 3. Displacement fields for a parabolic asperity for $\alpha < 1/2$(———); $\alpha = 1/2$(–·–·–·); $\alpha > 1/2$(----).

Fig. 3

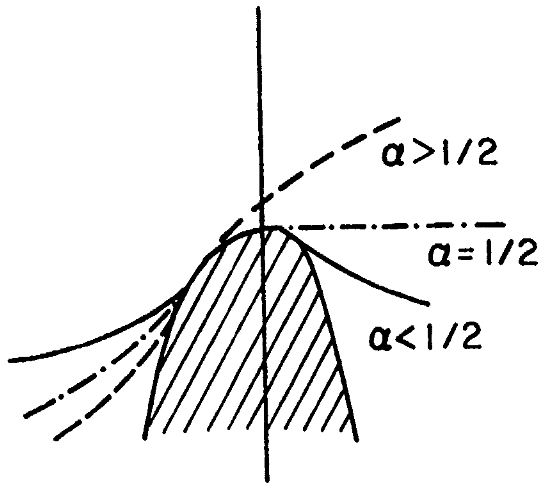

Figure 4. Normal traction fields over the contact interval for a parabolic asperity for $\alpha < 1/2$(——); $\alpha = 1/2$(-·-·-·); $\alpha > 1/2$(----). The interval (a_o,b_o) is only representative, it is not intended to imply that the same interval is correct for all α.

Fig. 4

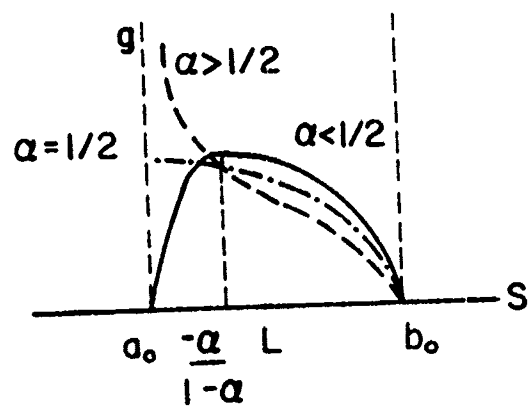

Figure 5. Displacement fields for a wedge indentor for $\alpha < 1/2$(———); $\alpha = 1/2$(-·-·-·); $\alpha > 1/2$(----).

Fig. 5

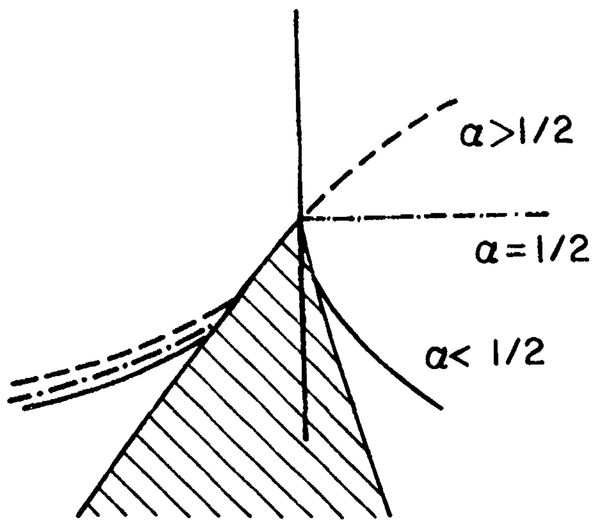

Figure 6. Normal traction fields over the contact interval for a wedge indentor for $\alpha < 1/2$(———); $\alpha = 1/2$(-·-·-·); $a > 1/2$(----). The interval (a_o, b_o) is only representative, it is not intended to imply that the same interval is correct for all α.

Fig. 6